back to the basics of NRA

the heavy lifting nobody* talks about

Gereon Kremer

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$$x \geq 2 \wedge x + y = 7 \wedge z > y$$

$$x \mapsto 2 \qquad y \mapsto 5 \qquad z \mapsto 6$$

nonlinear arithmetic

$$x^2 = 2$$

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let's open this box:

- what do $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{5 + 2 \cdot \sqrt{6}}$ actually mean?
- what happens in WolframAlpha?
- what do we need to do in `cvc5`?

canonical representation

- $\sqrt{2}, \sqrt{3}$
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- solve $x^2 y - xy^2 + x = 3$ under $x \mapsto \sqrt[3]{5}$
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$\Rightarrow$ is there a closed computational framework?

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important observations for `Real` from SMT-LIB: ignore `NTA`

- all input constants are in $\mathbb{Q}$
- all definable numbers for `*LRA*` are in $\mathbb{Q}$
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  "why would you?" – "$x > 0$" – "oh."

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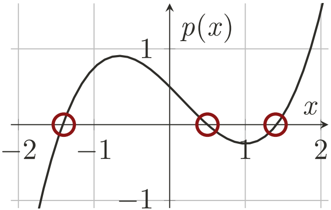

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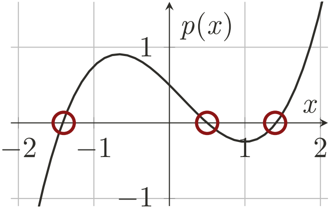

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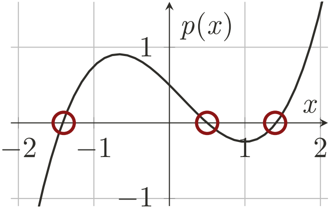

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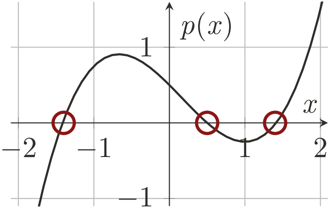

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with defining polynomial $p \in \mathbb{Z}[x]$, isolating interval $(l, u) \subset \mathbb{Q}$ and

$$\exists x^* \in (l, u).\, (p(x^*) = 0 \land \forall y \in (l, u).\, (y = x^* \lor p(y) \neq 0))$$

some examples

- $\sqrt{2}$: $(x^2 - 2, (1, 2))$
- $-\sqrt{2}$: $(x^2 - 2, (-2, -1))$
- $\sqrt[4]{8}$: $(x^4 - 8, (1, 2))$

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  $\sqrt{8 + 2 \cdot \sqrt{15}}$: $(x^4 - 16x^2 + 4, (3, 4))$
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- is there a canonical isolating interval?
  no. is $(1,2)$ better or worse than $(1.4,1.5)$ for $\sqrt{2}$?
  we can (and have to) refine the interval occasionally

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yes: $\gcd(p, q) = x^2 - 2$; use $(x^2 - 2, (1.5, 2.5))$; refine until contained

operations – more

$$a = (p_a, (l_a, u_a)) \overset{?}{<, >} b = (p_b, (l_b, u_b))$$

operations – more

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$a + b$, $a \cdot b$, $\ldots$
you can implement them... go read some papers.

what we actually want

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solve this instead: $q = 0 \land p_{\overline{x}} = 0$
this is well-studied in computer algebra!

system of equalities via variable elimination

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left to do: compute $roots(q^*) = \overline{r}$, check whether $q(\alpha, r) = 0$

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q may nullify and roots may be lost!
we can retain soundness, but comes with a cost. ($\rightarrow$ projection operators)

avoid nullification using Lazard

Lazard's lifting schema:

$$
\begin{array}{l}
\texttt{for} \;\; i = 0 \;\; \texttt{to} \;\; n \\
\qquad\qquad v_i = \arg\max_{v \in \mathbb{Z}} (x_i - \alpha_i) \;\; \texttt{divides} \;\; q \\
\qquad\qquad q = q / (x_i - \alpha_i)_i^v \\
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solves all our problems...?

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underlying issue:
if p_b factors over $\mathbb{Q}(a)$, $\mathbb{Q}(a, b) \ncong \mathbb{Z}[x_a, x_b]/\langle p_a, p_b\rangle$ — not even a field
general fix: factor p_b, use vanishing factor instead — factor over $\mathbb{Q}(\sqrt{2})$???

canonical representation – reprise

`cvc5` requires a canonical form for terms, also arithmetic terms
only reasonable canonical form:
collapse all numbers into a single real algebraic numbers.

$$\sqrt{11} \cdot \left(\sqrt[3]{3} + \sqrt{7} \right)$$

WolframAlpha:

$$\sqrt{\text{root of } x^6 - 462\,x^5 + 88\,935\,x^4 - 9\,154\,618\,x^3 + 499\,624\,125\,x^2 - 18\,371\,409\,672\,x + 197\,628\,258\,916 \text{ near } x = 183.829}$$

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cvc5:

```
<1*x^12 + (-462*x^10) + 88935*x^8 + (-9154618*x^6) + 499624125*x^4 + (-
18371409672*x^2) + 197628258916, (27/2, 55/4)>
```

conclusion

- nonlinear real arithmetic models are "special"
- representation is not (that) obvious
- arithmetic is not easy — not even conceptually
- some algebra is necessary

thank you for your attention!

nerd sniping

1. $q(\alpha_a, \alpha_b, c) = 0 \overset{?}{\Rightarrow} a \in \mathbb{Q}(b) \vee b \in \mathbb{Q}(a)$
2. can we construct $\mathcal{R}$?
3. why are there spurious roots after variable elimination?

nerd sniping – some answers

1. no; with $a = \sqrt{3 + \sqrt{3}}, b = \sqrt{3 - \sqrt{3}}$ although $a \notin \mathbb{Q}(b) \wedge b \notin \mathbb{Q}(a)$, $(a + b) \cdot c$ nullifies. the minimal polynomial is $x^4 - 6x^2 + 6$ irreducible over $\mathbb{Q}$ but factors into $(x + a)(x - a)(x^2 + x - 6)$ over $\mathbb{Q}(a) \cong \mathbb{Q}[a]/\langle a^4 - 6a^2 + 6\rangle$.
2. conceptually yes, practically no. for starters, every prime p yields a new field extension $\mathbb{Q}(\sqrt{p})$ not covered by any $\mathbb{Q}(\sqrt{n})$, $n < p$.
3. both resultants and Gröbner bases actually argue about complex roots. complex roots in the input may give rise to real roots in the output.